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What aspects can characterize the specialised knowledge of a mathematics teacher educator?

¿Qué aspectos caracterizan el conocimiento especializado del formador de profesorado de matemáticas?

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Abstract: This paper presents an exploratory qualitative study which was carried out by means of questionnaires for identifying some knowledge claimed by Mathematics Teacher Educators (MTEs) as important to be, or to become, a "good teacher educator". The study focused the analysis on some teacher educators' answers about MTE's specialised knowledge. Mathematics Teachers' Specialised Knowledge model (the so-called MTSK model) helped us to identify specific dimensions and domains of teacher educators' specialised knowledge, identified in some mathematics teacher educators' answers. From this information, by analysing similarities and differences and prevalent ideas, we raise hypotheses about some features that teacher educators believe as a part of MTE's specialised knowledge. Finally, we propose an adaptation of the MTSK model that can describe the teacher educator's specialised knowledge

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(Mathematics Teacher Educators' Specialised Knowledge model - MTESK model) by capturing and including in the model the main issues obtained in the study.

Keywords: *MTSK model · Mathematics Teacher Educator · Mathematics Teacher Educator· Specialised Knowledge*

Resumen: Este artículo presenta un estudio cualitativo exploratorio que se llevó a cabo mediante cuestionarios para identificar algunos conocimientos señalados como importantes para ser o llegar a ser un "buen formador de docentes" por los formadores de docentes de matemáticas (MTE, acrónimo en inglés). El estudio centró el análisis en las respuestas de algunos formadores de profesorado sobre el conocimiento especializado del MTE. El modelo de conocimiento especializado del profesorado de matemáticas (el llamado modelo MTSK) ayudó a identificar dimensiones y dominios específicos del conocimiento especializado de los formadores de profesorado, identificado a partir de las respuestas de los formadores de docentes de matemáticas. A partir de esta información, al analizar similitudes y diferencias e ideas prevalentes, planteamos hipótesis sobre algunas características que los formadores de docentes creen que son parte del conocimiento especializado del MTE. Finalmente, proponemos una adaptación del modelo MTSK que puede describir el conocimiento especializado del formador de profesores (modelo de Conocimiento Especializado de los Formadores de Profesores de Matemáticas - modelo MTESK) capturando e incluyendo en el modelo los principales aspectos obtenidos en el estudio.

Palabras clave: *modelo MTSK · formador de profesorado de matemáticas · profesorado de matemáticas · conocimiento especializado*

1 INTRODUCTION

Until a few years ago, the knowledge of Mathematics Teacher Educators (MTE) remained under-researched (Dinham, 2013), but in recent years researchers agreed in the need of more evidence, and, thus, this issue is gaining relevance in conferences (Llinares and Krainer, 2006; Ribeiro *et al.*, 2017, 2019) and in mathematics education journals and Books (Beswick and Goos, 2018, Goos and

Beswick, 2021). We aimed our research at a better understanding of MTEs' features identifying teacher educators' specialised knowledge. We asked MTEs about good features of a MTE. Thus, we collected information about what Mathematics Teacher Educators claimed as important to be, or to become, a "good teacher educator". In our analysis we focused on some aspects related to knowledge. A common way to define MTEs' knowledge is extending teachers' knowledge for teaching (Even *et al.*, 2018). The goal of our study is to generate hypotheses about how to modify the Mathematics Teachers' Specialised Knowledge model - MTSK (Carrillo-Yáñez *et al.*, 2018) to better describe and analyse MTE specialised knowledge.

In this paper, after presenting the theoretical framework, we describe the structure of the study and the main results. As a final step, from the data gathered, we raise hypotheses about some features that teacher educators claimed as important to be, or to become, a "good teacher educator" and then, we propose an adaptation of the MTSK model that can describe the teacher educators' specialised knowledge (Mathematics Teacher Educators' Specialised Knowledge – MTESK).

2 RATIONALE AND THEORETICAL FRAMEWORK

In this section we discuss the reasons for choosing the MTSK model as a descriptive and interpretative tool and present it in all its parts.

2.1 Mathematics Teacher Educators: knowledge and beliefs

A Mathematics Teacher Educator (MTE, MTEs in plural) is a professional figure in charge of educating mathematics teachers, who may be pre-service or in-service. An MTE can have different characteristics according to his/her background and to the contexts in which he/she works. "[...] mathematics educators include educators who facilitate the learning of mathematics, as well as educators who facilitate the teaching of (or learning-to-teach) mathematics" (Zaslavsky, 2009; p. 1). This is the reason why MTEs' knowledge and skills, and the objects of their teaching, greatly differ. Research has focused on these different profiles of MTEs, generating several denominations, as teacher trainer (Palhares *et al.*, 2009), professional development facilitator (Linder, 2011; Zaslavsky and Leikin, 2004), didactician (Jaworski, 2012; Coles, 2014) or mentor (Halai, 1998), among others, depending on their content they teach (mathematics or mathematics didactics), the teachers they train (in-serve or pre-service), whether they are also researchers or not, and the intensity of their training (academic degrees, internships, specific PD courses, etc.). Due to the exploratory approach followed in this paper, we wanted to consider all the possible profiles, therefore, rather than focusing on one clearly defined term (as mentors or didacticians) we opted for the broader MTE, as in Beswick and Chapman (2012), Escudero-Ávila *et al.* (2021) or Pascual *et al.* (2021).

Researchers all over the world are investigating MTE's knowledge, work, education, learning and growing (Jaworski and Wood, 2008; Krainer *et al.*, 2014; Even *et al.*, 2018). Some studies highlight the pivotal and strict connection between the MTE's knowledge and teachers' knowledge (Jaworski, 2008; Even *et al.*, 2018; Escudero-Ávila *et al.*, 2021). Thus, some authors aimed at describing the relationship between MTEs' and teachers' knowledge and framing how MTEs broaden and develop their theoretical knowledge by designing and carrying out professional development paths (Jaworski, 2001; Zaslavsky *et al.*, 2003; Peled and Hershkovitz, 2004; Ozmantar and Agac, 2021). Researchers have analysed specific activities carried out by MTEs, highlighting that MTEs' educational goals and practices are influenced by their mathematical knowledge, pedagogical knowledge, and beliefs.

We find different frameworks describing mathematics teacher's specialised knowledge, as the Mathematics Knowledge for Teaching – MKT (Ball *et al.*, 2008; Hill *et al.*, 2008), the Knowledge Quartet (Rowland *et al.*, 2009) and the Mathematics Teachers' Specialised Knowledge model – MTSK (Carrillo-Yáñez *et al.*, 2018). In our study we use the MTSK model to investigate MTEs' specialised knowledge and we propose hypotheses on how to "extend" this model, for capturing features that characterize MTEs' specialised knowledge. As Beswick and Goos (2018) stressed, research on MTEs' knowledge can be extended also to their own beliefs and reflections about what they teach. We have chosen the MTSK model for this analysis because this model explicitly takes in consideration teachers' beliefs. As underlined by Carrillo-Yáñez *et al.* (2018), teachers' beliefs about mathematics and its teaching/learning are incorporated within all domains of the model and permeate them, as they give meaning to their actions.

2.2 MTSK model

Our research started from the possible use of MTSK model (Carrillo-Yáñez *et al.*, 2018) to describe MTEs' specialised knowledge. Often in literature, to define MTEs' knowledge researchers use extensions of interpretative tools used for studying teachers' knowledge for teaching. Recently, the MTSK has started to be used as a basis model for conceptualizing MTEs' knowledge in two different ways. The first one (Pascual *et al.*, 2019, 2021), uses lesson observations to extract information about MTEs' conceptions and practices. The second one (Escudero-Ávila *et al.*, 2021) is grounded on what research revealed as relevant knowledge for teachers to induce MTE's knowledge.

In this paper we use a third approach, based on MTEs' declared statements about what characterises MTE's knowledge. We can then consider the information gathered in these statements as beliefs about the specialist knowledge of MTEs. This paper is not for discussing the definition of belief, but it is necessary to explain, even briefly, our premises. As beliefs were defined by Philipp as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (Philipp, 2007; p. 259), researchers do not agree about both the definition and the role of beliefs in teachers' practice, thus, "the relationship between belief and practice has been a philosophical enquiry with, inevitably, conjectural outcomes [...], whilst others have sought confirmatory evidence" (Andrews and Hatch, 2000; p. 31). Although the influence of teachers' beliefs on their practice seems to be acceptable (Clark *et al.*, 2014), it is not clear how it is produced (Schoenfeld, 2011). A usual way to embed beliefs into a more general frame are conceptions, intended to gather desirable instructional goals for teachers together with their "own role in teaching, the students' role, appropriate classroom activities, desirable instructional approaches and emphases, legitimate mathematical procedures, and acceptable outcomes of instruction" (Thompson, 1992; p. 135). In Philipp's words, conceptions are mental structures which encompass "beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (Philipp, 2007; p. 259).

In our research we opted for a methodological approach based on qualitative investigations, by asking MTEs about the desirable features defining a good MTE, thus, since we are analysing MTEs' answers, it becomes impossible to differentiate between beliefs and conceptions. What we obtained were MTEs opinions involving their own knowledge, beliefs, preferences, etc. Therefore, we are within Philipp's definition of conceptions, even when we could assume beliefs are expressed. Given that impossibility, in order to gain conceptual clarity, we will refer to beliefs and conceptions through all the paper, without retaking the discussion about their nature. The intimate connection in MTEs' answers among beliefs, conceptions and knowledge also justifies our election of the MTSK model as theoretical framework for analysing the retrieved information about MTEs' knowledge. Beliefs and conceptions about mathematics and its teaching and learning are explicitly present in the core of the MTSK model, constituting themselves a domain, as the content and the pedagogical content ones. This domain permeates the other two ones, and this is the reason why it is mapped as in the centre of the model (Figure 1).

The starting point of the MTSK model was the difficulty in fixing certain overlaps between common and specialised knowledge in the MTK (Flores *et al.*, 2013). Therefore, one initial assumption in the MTSK is extending the idea of specialization to all the knowledge domains and considering that knowledge for teaching which is conditioned by mathematics. That is, general pedagogical knowledge or skills, for instance, those related to classroom management, are not usually considered within that model. What makes mathematics teacher's knowledge specialised is not so much what mathematics teachers know (which might indeed differ from other professionals), but how mathematics teachers know. In authors' words: "the specificity of the teacher's knowledge in relation to mathematics teaching affects both SMK and PCK together, and as such cannot be considered a sub-domain of either" (Carrillo-Yáñez *et al.*, 2018; p. 4).

On the one side of Figure 1, we see the Mathematical Knowledge (MK) domain encompasses knowledge of the connections between concepts, structuring ideas, procedure reasoning, proving, and different ways of proceeding in mathematics, also considering the knowledge of mathematical language. Within the MK there are three subdomains. Knowledge of Topic (KoT) includes the knowledge of concepts, properties, procedures, facts, rules, theorems, etc. Knowledge of the Structure of Mathematics (KSM) describes teacher's knowledge about mathematics and its structure: how mathematical items are connected both in vertical (same idea with more or less complexity) and in horizontal (connections between mathematical objects). Knowledge of Practices in Mathematics (KPM) comprises two approaches: one is general, about how mathematics is developed (proof, argumentation, counterexamples, etc.) and the other one is about the peculiarities of a specific topic (e.g., approaches to proof a result within probability theory).

On the other side, the Pedagogical Content Knowledge (PCK) refers to teacher's knowledge of mathematical content as an object of teaching and learning. The PCK is divided into three subdomains. Knowledge of Mathematics Teaching (KMT) includes the knowledge required for designing activities, choosing strategies, selecting resources, and studying teaching approaches. Knowledge of Features of Learning Mathematics (KFLM) includes teacher's awareness about how students' construct their mathematical knowledge, and what can be their difficulties, in general and regarding a specific topic. Knowledge of Mathematics Learning Standards (KMLS) includes the knowledge of official and non-official curricular guidelines (as standards from national ministries or from the National Council of Teachers of Mathematics [NCTM]), and the contents to be taught at any educational level.

The MTSK model has been widely used in analysing what knowledge is mobilized by mathematics teachers but also how that knowledge is influenced by teachers' belief and conceptions (Red Iberoamericana MTSK, redmtsk.com). The central position of beliefs and conceptions of mathematics and its teaching and learning underlines the need of considering them as a domain in the model, but playing a different role that MK or PCK, since beliefs and conceptions are intimately related, underlying and permeating every evidence of the other two domains. Therefore, there is research line about methodologies for determining how this relationship between MK or PCK and beliefs and conceptions is organized (Aguilar-González *et al.*, 2018, 2019).

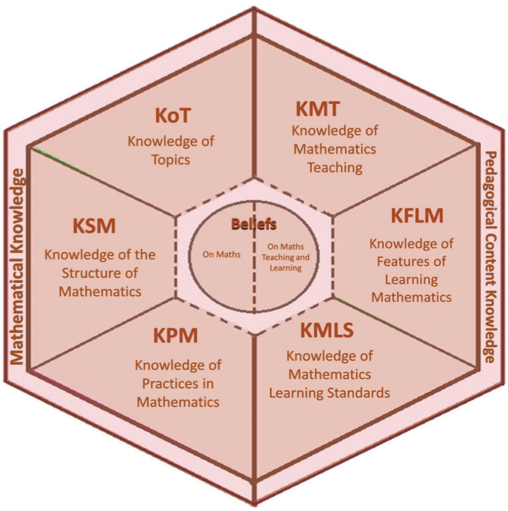

Figure 1. The MTSK model (Carrillo-Yáñez *et al.*, 2018, p. 6).

Understanding the beliefs that underpin the practice of MTEs must be at least as important as understanding those influencing the work of mathematics teachers (Beswick and Goos, 2018). As stressed before, the fact that the MTSK model makes central and explicit the reference to teacher beliefs, makes this model suitable for the purposes of our study because we aim to investigate the features that some

MTEs claim as important to become MTE. Therefore, we collect data on their beliefs about their mathematical knowledge and the aspects on mathematics teaching and learning related to teachers and students. The question was: will the model provide us with all the necessary keys to identify the characteristics of the specialised knowledge of teacher educators declared by MTEs? Therefore, we started from the theoretical lenses given by the MTSK model. We are aware that there would be aspects characterizing MTEs' knowledge different from those in the case of MT, but our purpose is not to define the content of the model domains that already exist, but to see if other domains or dimensions can be added. We analyse MTEs' beliefs on their specialised knowledge and also their knowledge and above all their reflections on teachers specialised knowledge. In this perspective, we are aligned with Beswick and Chapman (2012) who suggested that MTE's knowledge can be thought of as a kind of meta-knowledge including some of the knowledge that mathematics teachers require. MTEs work with and teach to different groups: researchers, other MTEs, teachers, and students. MTEs often act as brokers belonging to more than one community (Rasmussen *et al.*, 2009). They have a key role in the processes of meta-didactical transposition carried out by MTEs and teachers when they are engaged in teachers' education activities (Aldon *et al.*, 2013; Robutti, 2018). In this process they are teachers, but also learners (Krainer *et al.*, 2014). In the educational project in which they are involved, MTEs use and develop their mathematical knowledge for teaching teachers also by working together and reflecting in their community of practice (Masingila *et al.*, 2018; Masingila *et al.*, 2019). Our study was not concerned with studying the processes that take place during the teacher education programmes, but in describing the aspects that characterise the specialised knowledge of MTEs we highlighted aspects that can be identified as related to the meta-knowledge that can be developed by MTEs. In particular, in this paper we show some aspects of MTEs' knowledge and meta-knowledge that MTEs claim to be important.

3. THE STUDY

We develop our study within two different European countries (Italy and Spain) sharing some characteristics but also differing about the educational paths to become MTE. In Italy and Spain there are different administration issues and didactic traditions that produce different ways of becoming a MTE, especially for those who are teachers at primary or secondary level. Thus, in Italy were seminars and education projects developed by researchers and national institutions, and this has been a way for most of the Italian MTEs to train as MTE. In Spain it is also possible to find such seminars but much less organized and systematized, so that they do not have the same importance for becoming MTE. For those MTEs being university researchers, the trajectories seem to be more similar, also like most European countries: starting with a Ph.D. in mathematics education and, simultaneously or immediately after, getting enrolled as MTE. In this research, we consider MTEs who oversee educating (pre-service or in-service) mathematics teachers.

This group of teachers is certainly not representative of Italian or Spanish teachers. But representativity was not the aim of the study, instead, in our exploratory study our goal is to find some evidence of dimensions or domains that can be added to MTSK model. This can be the object in the future of a quantitative study on significant and representative samples. What we present here is a study that allows us to highlight aspects and generate hypotheses on the characteristics that some MTEs think are important for an MTE. In this qualitative study we only need evidence to say that these aspects exist. In future research we will have to investigate with other methodological tools how widespread they are and how important they are. We want to identify some characteristics that MTEs claim to be important in order to be, or to become, a good teacher educator in both countries.

3.1 Aims of the study and research questions

We asked MTEs what knowledge and skills are needed to become an MTE and what characteristics a good MTE should have. We chose to ask MTEs not only what knowledge is needed but, more generally, what characteristics a MTE thinks are needed to be a good MTE because in this way we opened up the possibility for the participating MTEs to tell us features that they think to be important in order to be, or to become, a "good teacher educator". In our analysis we focused on the aspects linked to mathematical knowledge.

Since we ask MTEs what they think is needed to be a good MTE, in their answers we can only identify their beliefs and reflections. As discussed before, we use the MTSK model because its structure takes explicitly into account the beliefs on various types of specialised mathematical knowledge for teaching.

Using the theoretical lenses given by MTSK model we analysed MTEs' answers and we noticed that some of the domain of MTSK model could be extended and adapted in order to better describe MTE specialised knowledge. Therefore, starting from a preliminary analysis of a few answers, we focused our attention on aspects that could expand some domains of the MTSK model. Our study aimed to generate hypotheses about how to modify the MTSK model to be better used for the description and analysis of some features of MTE specialised knowledge.

3.2 Methodology

Our research started with a pilot study to validate the questionnaire and to test the MTSK model as an interpretative tool to label specific dimensions and domains of teacher educators' specialised knowledge, identified in some mathematics teacher educators' answers. The data gathered by the questionnaire were analysed through an inductive content analysis (Patton, 2002). A top-down coding was performed (from model to the data): we used the theoretical lenses provided by the MTSK to highlight beliefs on different domains and subdomains of specialised mathematical knowledge. We developed a parallel analysis and then exchanged the extracted data to cross-check consistency of classifications: each of us independently read the MTEs' answers and he/she identified knowledge features by using the MTSK model, then we compared the results obtained. This preliminary study showed us that we can identify in MTEs' answers references to different domains of the MTSK model. Then, by means of a bottom-up strategy (from data to model), we carried out an analysis on the entire group of informants to identify recurrent items that emerged from the MTEs' answers that can be used to extend the MTSK model.

3.2.1 Informants

The informants selected for our study were *N*=13 MTEs, 10 from Italy and 3 from Spain (respectively identified with codes TI or TS, plus a number). Since this study is a qualitative exploratory analysis, the unbalance of nationalities does not constitute a severe limitation in the scope; moreover, we will remark that results do not show very significant differences by country.

As underlined before, because we aimed at raising hypotheses about which could be the characteristics an MTE claims to be important in order to be o become a "good teacher educator", the selection of the informants was based on the ease of access to the MTEs. Thus, we aimed at considering a variety of profiles in terms of background, occupation (including research profile when applicable) and teaching experience, ranging from 30 to 60 years old.

In terms of background most of the participants are graduated in mathematics, but 3 are graduated in primary education. Regarding the occupation, the majority of the sample (8) share a tenure as MTE (at university or other institutions) with being in-service primary or secondary teacher, being the rest university professors. In terms of the trainees, 9 MTEs usually train both in-service and pre-service teachers, 2 teach only pre-service and 2 only in-service. Finally, 5 MTEs are also involved in tutoring prospective primary or secondary teachers' internships at schools. Additionally, we must underline that all the involved MTEs also teach mathematics either at university or school level. This distribution covers the two main categories defined in Koster *et al*. (1998) showing a great overlapping between the so-called school-based and university-based teacher educators, and it hampers analysing separately pre-service or in-service MTEs' answers.

Further than common features, the informants show very diverse trajectories. Nine MTEs have a long, varied trajectory in teaching mathematics teachers, with combinations of educational levels and activities (primary, secondary or university; pre-service and/or in-service) whereas three of them started more recently to act as MTE at university level. Some of the MTEs have recently obtained a PhD degree in mathematics education, others hold a PhD degree in mathematics or in education, and some do not hold a PhD degree. One of university professors is a researcher in applied mathematics, but involved in teacher education, being the rest working within mathematics education research. One MTE has a high-rank experience as educational stakeholder in the administration.

3.3.2 Instrument

A questionnaire was administered by written form or structured oral interview to the participants. It was composed of an introductory section in which they were asked about his/her background, current position and how many years of experience as MTE they had. The questionnaire consisted of two open-ended questions (Cohen *et al*., 2007):

> Question 1: 'What knowledge and skills do you think someone needs to become a Mathematics Teacher Educator?'
> Question 2: 'What characteristics do you think a "good" MTE should have?'

Questions were originally designed in English, then authors translated them to Italian and Spanish, and, after discussing the validation of the translation, they were administered in the respective mother tongue (Italian or Spanish) corresponding to each interviewee. Researchers did not interact during the interviews (they only ask the two questions together – orally or in writing) and they gave total freedom of expression without limiting time or length for the answers.

4 PILOT STUDY

As starting point of our research, we piloted a brief study by collecting and analysing three interviews: two MTEs from Italy (TI_01 and TI_02) and one from Spain (TS_01). This paragraph presents some excerpts from these MTEs' answers. By means of the MTSK model theoretical lenses we identified sentences that deal with MTE's beliefs about Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK) different domains.

The MTEs agreed that a good MTE should have a strong mathematical knowledge: the following sentences can be labelled as beliefs on Knowledge of Topics (KoT).

> TI_01: A good MTE should have a broad knowledge that embraces several disciplines. First of all, he/she should have a strong mathematical content knowledge. I believe that it is somehow naïve pretending to train mathematics teachers without a suitable mathematical background.
> TS_01: A good MTE has to combine different elements. He/she must have a mastery of mathematics. Either because he/she studied it in the previous background (which seems that the most recommendable background is a degree in mathematics), or because he/she shows the ability to revise the mathematics and achieve a mastery. Once the mastery is assumed (concepts what he/she has to teach).

We found beliefs evidence on Knowledge of the Structure of Mathematics (KSM), e.g., TI_01 recognized that an awareness of the historical development of mathematics and its foundational cornerstones opens the way to reflections regarding teaching-learning processes in mathematics:

> TI_01. Besides the basic topics that a graduate in mathematics encounters in his university studies, an MTE should develop an epistemological and historical outlook on the discipline. The awareness of the historical development of mathematics and its foundational cornerstones opens the way to reflections regarding the teaching-learning processes in mathematics.

We also found evidence of beliefs on Pedagogical Content Knowledge, particularly, about Knowledge of Mathematics Teaching (KMT) for students. Some MTEs expressed their beliefs about recognising and handling class diversity: they underlined the existence of a great variety of prospective math teachers, so that an MTE has to understand and adapt to this diversity. For example, TS_01 says:

> TS_01: He/she also has to have the ability to control the class, to see what the profile of the trained students is, to understand that there is a variety of students and he/she has to model their learning by using different activities that are achievable for all of them: not so low-level that can be boring for students mastering mathematics, and not so high-level that make students with low mastery could get disconnected of the course. [...] What I think the most about it for improving my work as TE is learning to value the diversity in the class. That is, how students with notably different backgrounds and notably different mathematical knowledge could become levelled in order to teach mathematics with the required rigor. I think I have been learning this step by step.

An emerging aspect from these first interviews is that MTEs claim the importance of knowing how to teach teachers and how they can learn. Therefore, we added to the MTSK model, the beliefs about needed Knowledge to teach teachers.

> TI_02. I believe that a good teacher educator is able to balance between theory and practice. Between principles and examples. Sometimes one can adopt a top-down approach (from theory to examples), other times a bottom-up approach (from examples, provided by the teachers themselves, to theory). [...] teachers love to be listened to. To feel that they are cared about. They like to discuss with their colleagues, and they like to engage in discussions with the teacher educator. So, one who provides opportunity for talking does not only allow teachers to 'feel good', but also enhances interpersonal skills, deeper understanding of the subject and a sense of connection with the class.
>
> TS_01: he/she also has to demonstrate the capacity to transmit that knowledge to the students (pre-service teachers), in order to do that he/she has to acquire several learning and methodological strategies, so that what he/she knows can be also know (until a certain degree) by the students (pre-service teachers)

Another aspect that has emerged as MTEs' belief about the knowledge on teaching and learning mathematics, concerns the knowledge of research in mathematics education and the importance of attending conferences.

> TI_01: Mathematics education literacy is fundamental to become an MTE [...] mathematics education provides powerful tools to design appropriate mathematics curricula, assessment approaches and teaching methods and strategies.

5 Results and analysis

Starting from the data and analysis carried out in the pilot study, we enlarged the number of informants and, hence, the results presented below include all the answers. We used the MTSK model interpretive lens, and we also considered the aspects that emerged in the pilot study such as teachers' knowledge of teaching-learning or the importance of knowledge of educational research. We followed the same process of analysis of the answers that we carried out in the pilot study.

5.1 Question 1: results and analysis

Answers to Question 1 ('What knowledge and skills do you think someone needs to become a mathematics TE?') were broad and extensive. In this paper we focus on MTE's sentences linked to PCK and MK. Ten MTEs answered by pointing out different versions of PCK or didactic approaches, that is, answers about methodological aspects, adaptation of the knowledge to the trainees, laboratory methodology to put theory into practice, teaching about resources and materials, etc. Eight MTEs underlined the importance of MK as one required knowledge for an MTE. Knowledge of technological tools and of the historical development of mathematics were pointed out once each. Additionally, experience as a teacher appeared in four of the answers. Knowledge of evolutionary psychology also appeared three times, as well as the knowledge about curriculum and curricular guidelines. Two MTEs answered that classroom management is an important MTE skill. Special attention must be given to knowledge of educational research, including knowledge of examples and tasks, because it was mentioned 5 times.

The data collected show that there is a clear combination of MK and PCK, appearing the later under different varieties (methodology, curriculum, etc.). It is important to point out how university researchers unanimously agree in the didactics of mathematics. On the other hand, mathematics secondary teachers reach a high degree of consensus in balancing mathematics and pedagogical content knowledge. What is interesting for the goal of this research is how different MTE profiles have different degrees of consensus regarding the most cited Knowledge. It is also interesting how some of beliefs about PCK are aligned with the movement from a mathematical PCK and the so-called school mathematics PCK (Chick and Beswick, 2013), characterized for providing teachers examples and contexts of the use of mathematics, so that they are able to establish connections for a better interaction with students when they teach.

MTEs quote also different aspects related to psychology. Laski *et al.* (2013) showed that MTEs acknowledge the need of incorporating issues about cognition (particularly, evolutionary psychology) in their teaching activity, despite these authors found a great diversity in how MTEs treated and considered knowledge about cognition. Our findings are consistent with that diversity of approaches. Some of the answers agreed on the importance of curriculum/standards knowledge in Tes' activity, which was previously analysed by Chauvot (2008), who encouraged researchers to study themselves during their transition from teachers to MTE for a better understanding of the transformation process.

5.2 Question 2: results and analysis

Question 2 ('What characteristics do you think a "good" MTE should have?') had 2 high consensus answers and others with less prevalence. 8 MTEs underlined different facets of active and adaptive learning (learning by doing, dialoguing, group working and managing, putting students into a real situation, being down to earth, etc.).

4 MTEs considered keeping updated about the subject, or keeping curiosity alive, as well as other 4 MTEs pointed out being aware of educational research in didactics, including here 2 answers that addressed how to promote discovery innovative paths among teachers, without an explicit mention to educational research. 3 MTEs explicitly quoted the participation in research and training seminars. 2 MTEs answered having knowledge of the didactics of mathematics, and another two being in contact with other colleagues (which could be understood as a specific communication skill). Being aware of one's beliefs and having experience at school level as a teacher were also pointed out once each.

In MTEs' answer we can notice different interpretations that they have about what is research at school. This could explain 4 answers about being updated and 4 ones about being aware of research. We see that those pointing out being updated are secondary and primary teachers, while those pointing out being aware of educational research are 50/50 researchers in mathematics education and secondary teachers.

A highly considerable aspect that emerged in most of the answers is the reference to the knowledge of mathematics education research. This knowledge refers both to teaching and learning and, thus, it could be considered a cross-domains knowledge that we labelled KoMER (Knowledge of Mathematics Education Research). KoMER could be also part of teacher's specialised knowledge, but it becomes very relevant for MTEs. MTEs can play the role of brokers between teachers' and researchers' communities (Aldon *et al.*, 2013). Bridging theoretical research and practice through the mediation of theory constitutes a pivotal role for MTEs; as highlighted in Lin *et al.* (2018), often educators introduce theory to teachers and the tension between theory and practice poses challenges to teacher educators. In fact, 11 MTEs roundly referred to the importance of knowing the research from different points of view. The majority of them explicitly mentioned Knowledge of Mathematics Education Research as a knowledge and skills that MTE should have. Most MTEs refer to their own personal paths, claiming to have deeply studied mathematics education research, both as regards theoretical research and what concerns methodological issues (mainly concerning laboratory activities).

Many MTEs stressed the importance of taking part in mixed groups of teachers and researchers. 6 out of 8 Italian MTEs said they became MTEs thanks to the participation in mixed research groups represented by researchers and teachers. They declared that the participation in these groups allowed them to strengthen their mathematical knowledge and to be updated on research on mathematics education. This is tuned with Goss' assertion: "Mathematics teacher educators' beliefs about teaching and learning are likely to be influenced by theoretical studies and research" (Goos, 2009; p. 213). It is significant to notice that all MTEs who said that the participation in these groups has increased and consolidated their mathematical knowledge are primary teachers. All the Italian MTEs who participated in the research groups, described them as a community of inquire (Jaworski, 2006) which unfolds in terms of critical thinking, questioning, doubting and bringing new points of view, they had the opportunity to discuss peers and researchers both regarding theoretical issues and didactic practices. In many MTEs' statement it emerges that the participation in these research groups has fostered reflections on the answers between theoretical perspective of the constructs of reflection and actions and the actual impact that can be had on teaching practices. On the other hand, the lack of such an institutionalised structure with mathematics teachers' professional development in Spain is clearly evidence when none of the Spanish MTEs referred to such a way for becoming MTE.

5.3 Analysis of common issues to both questions

From the analysis of the MTEs' answers there are meta-aspects of MTE knowledge that emerged. Keeping the focus on mathematical knowledge and pedagogical content knowledge, we report in this paragraph some of the sentences that we have identified as related to MTE knowledge and reflections on teachers' MK and PCK. The meta-aspects of MTE knowledge can be noticed in all domains of MTE knowledge. In fact, since the pilot study, we identified some sentences concerning MTEs' knowledge and reflections not only on their knowledge (TI_01), but also on teachers' specialised knowledge (TS_02; TS_03).

> TI_01. A meta-skill that a TE should develop is the recognition and awareness of what we could term as his/her internal epistemology, made up of beliefs, convictions, expectations, standpoints regarding mathematics and the cognitive, psychological and social aspects that characterize teaching and learning processes. I strongly believe that the awareness of his/her own internal epistemology allows a TE to pinpoint the one of his trainees and provide them with a broad range of interpretations that opens a "view from above" of the complexity within the teaching-learning processes of mathematics.
> TS_02: In the initial training you have to consider what is ideally good and what is the reality they are going to find at schools. In the in-service training, since teachers attend voluntarily, they have a big interest in knowing. So that you have a much easier task, since they use to have a clear idea about what they want and what they need. Therefore, you have to have knowledge about your topic but, especially, you have to know what your audience is. [...] On the other hand, sometimes these teachers have not studied mathematics, but engineering, then I had to work with them also about mathematical contents, because they had studied applied mathematics, and this is not the same as knowing why mathematics is how they are. [...] I have seen teachers knowing a lot of mathematics, but they are not able to connect with students.
> TS_03: A teacher educator needs a good mastery of the subject, as also a teacher needs a wide domain of the subject.

An important issue that arose in both questions is communication, together with active/adaptive learning skills in the second question. They are not properly mathematics specialised knowledge, but MTE quote them as competences of any teacher educator, no matter the topic we are referring to. Anyway, it is clear the need of a further exploration about the role, particularly, of communication. Within this context, it is strictly linked to mathematical (and not only general) communication, and it becomes clear that it characterizes MTE's activity. Therefore, deeper investigation about the role of communication for MTEs should be developed in order to complete its role within a meta-knowledge model. In fact, the role of the so-called meta-communication in MTE's activity have been recently pointed out in some authors' self-reflections about their professional transit from mathematics teachers to MTEs (Brown *et al.*, 2018).

The exploratory vocation of this work prevents us from obtaining generalizations, but there is a significantly arising issue: there are two different views about what knowledge is needed depending on participants' background. Thus, MTEs having a background as primary teachers mention mathematical knowledge much more than mathematics didactics, and, conversely, MTEs having a background in mathematics (secondary and university teachers) claim for a deeper PCK. When we look at these two paired results, they reveal that MTEs make a kind of self-assessment about how their training enabled them for the profession. Hence, primary teachers discover their need of deepening into MK, not into PCK, probably because their training was much more focused on didactics, and they realised about the need of MK. As Zaskis and Zaskis (2011) stated MTEs' MK "is not the kernel but the means" (p. 262) to develop a high-level awareness of their profession, these authors also wondered whether it is possible to achieve that level without being exposed to advanced mathematics. We consider that the answer provided by primary teacher MTEs points out in the same direction, of course without determining how 'advance' that mathematics should be. On the other hand, MTEs having a mathematical background underlined the need of having PCK, while didactical training is generally out of mathematics bachelor's degree syllabi. In other words, it seems that during the process of becoming a MTE, each group missed the training they did not have, either PCK or MK.

6 From MTSK model to MTESK model

In our study we started using MTSK model (Carrillo-Yáñez *et al.*, 2018) to identify features of knowledge that emerged from MTEs' answers about what is important in order to be, or to become, a "good teacher educator". In the analysis of MTEs' answers, we noticed that there are set of ideas characterising the MTEs knowledge, as they are the knowledge of mathematical education research and the knowledge about teaching to teachers. MTEs are often teachers or researchers but with features transforming them into experts in teacher education. Therefore, as some research underlined (see par. 2), there is not only knowledge of the subject or knowledge of the teaching and learning at school, but also knowledge concerning reflections on practices and activities with trained teachers. Moreover, we argue that MTE meta-character has to be added to the model because MTEs know and reflect on the different mathematical knowledge that a teacher needs when he/she acts as a trainee.

Another important aspect of MTE that arises is linked to knowledge of the educational research, the subdomain of the model that we labelled the Knowledge of Mathematics Education Research (KoMER). In fact, in MTEs' answers we found reference (more or less explicit) to the fact that knowing research results or joining research projects can help an MTE in his/her work during professional development courses. As it has been specified, many Italian MTEs are teacher-researchers and this fact has inevitably affected this result. Several MTEs, both Italian and Spanish, have declared that they have followed master's degrees or innovation/research projects for MTEs, and this fact is in line with the international literature in which it is underlined the importance of special courses for providers of professional development activities for mathematics teachers or programmes enhancing growth of mathematics teacher educators through their practice.

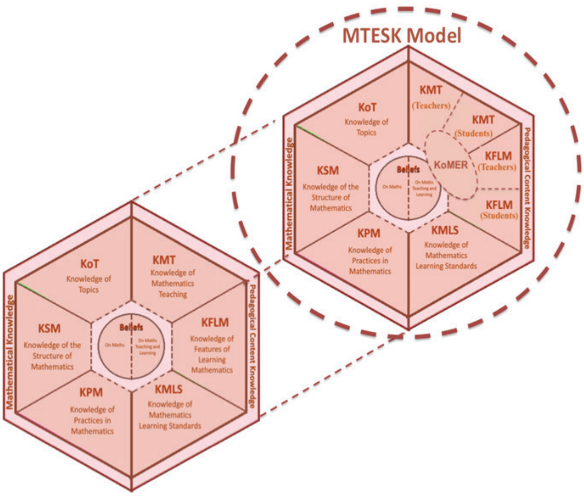

Figure 2. The MTESK Model.

Figure 2 shows a representation of the Mathematics Teacher Educator Specialised Knowledge model (MTESK model) we propose after the analysis of the data collected. The MTESK model shares the hexagon structure and some domains with the MTSK, but MTESK presents also specific features, as argued in this research: two of the PCK domains, KMT and KFLM, have been adapted by unfolding them into two parts. One part is linked to teaching-learning processes regarding students and the other part is linked to teaching-learning processes regarding mathematics teachers as trainees. Additionally, the importance of the Knowledge of Mathematics Education Research (KoMER), which is also mentioned in the MTSK model in the part dedicated to PCK, has been more emphasized by explicitly embedding another subdomain. Additionally, the meta-feature of MTE knowledge is stressed by representing the MTESK model projected on the MTSK. Finally, we must remark that the correspondence in the structure between some of the MTSK and MTESK domains does not necessarily mean equality, for instance, when considering MK, the knowledge that a mathematics teacher needs is not the same as the one needed by an MTE (Escudero-Ávila *et al.*, 2021), and probably the latter contains the former. Further details about the PCK domain can be found in Ferretti *et al.* (2021).

In conclusion, we can summarize the results of our study in this way: by using information from an open-ended questionnaire and the MTSK Model theoretical lenses we identified some specific features that MTEs declare to be important and to characterize MTEs' specialised Knowledge. Data provided some evidence that allowed us to generate hypotheses on how we could adapt MTSK model for MTE knowledge. This first extension of the model highlights the meta-features of MTEs' knowledge and it identifies also domains to be added to MTSK Model. Following the perspective in which the MTSK model was built, this could be the first step in the definition of the model and we are fully aware that new aspects will emerge from other studies. These aspects should be investigated by means of the analysis of MTE during their work, and not only by considering MTEs' opinions as we do in our study. The MTEs' specialised knowledge will be analysed when they occur in specific contexts as is the case of specialist knowledge of teachers (Scheiner *et al.*, 2017). We are concerned about there could be strong differences between the beliefs that MTEs declared and those guiding their behaviour when working: as shown in studies carried out on student and teacher beliefs, it can happen that there is a mismatch between teacher espoused beliefs and his/her practice (Malara and Zan, 2002). The results of our study, however, do not concern this possible discrepancy because our goal is only to generate hypotheses on possible domains to be expanded and modified in the MTSK Model.

7 FINAL REMARKS

This study has been conducted as exploratory, aiming at defining a basis for supporting a deeper understanding of MTEs' knowledge. As underlie by Even *et al.* (2018), the common feature of the majority of shared constructs, models and definitions of mathematics teacher educators' knowledge is developed by extending the teachers specialised knowledge, despite these extensions should consider that contents do not necessarily agree (Escudero-Ávila *et al.*, 2021). We modelled our ideas from the MTSK model, which was found to be the most suitable model for our purposes: i.e., to collect information to extend MTSK model for describing how MTEs declared that can be identified as MTE's specialized knowledge. Since we are working with declared information and not observations, considering MTE's beliefs intertwined with knowledge becomes a crucial issue. In this paper we only show the extended domains and dimensions that emerged from our first exploratory study, but of course we know that this is only one step in the construction of a complete MTESK model. Here we show only evidence that made us able to generate hypotheses on how to modify the MTSK model to better highlight some characteristics of MTEs' specialized knowledge. In particular, we highlighted that in the answers of the MTEs a "meta" dimension emerges that characterises a good TE for them and therefore we extended the MTSK model by highlighting this aspect. Furthermore, in the domain of PCK we have highlighted how knowledge related to mathematics educational research has an explicit place and the knowledge on how to teach mathematics to teachers (Fig. 2). In the future other aspects may emerge and this could lead to extending or modifying the identified domains. With this purpose, a larger scale study is going to be conducted with more MTEs with different background. Moreover, in future research we are going to determine if the observed similarities and differences between Spanish and Italian MTEs', fully aware that culture permeates all aspects of both mathematical practices and mathematics education practices (Bartolini Bussi and Martignone, 2013). In this regard, we must acknowledge that the diverse features of MTEs in Italy and in Spain are implicitly biassing the results of the study, and some of them could not hold in countries with different traditions.

A second future line will move from knowledge to other more general aspects, including pedagogical skills not necessarily related to mathematics, like communication or empathy. Nevertheless, we believe these issues must be considered under a mathematical lens, due to the vast literature about affective domain in teaching and learning mathematics, and the importance of communication as a mathematical process (NCTM, 2000), which is strongly related to processes of social construction of knowledge during initial or continuous teacher training (Ferretti, 2020; Muñiz-Rodríguez *et al*., 2020).

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